The mathematics of Luca Pacioli: appropriation, not plagiarism

Luca Pacioli (c. 1447–1517) conference
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Claims of plagiarism
Context: six phases towards the algebra textbook
Pacioli at the start of algebraic theory
Pacioli’s contributions to mathematics
Claims of plagiarism (1)

- The third part of *De divina proportione* (1509) is translated from Piero della Francesca’s *De quinque corporibus regularibus*.

- Giorgio Vasari, *Lives of the most eminent painters, sculptors & architects*, vol 3., p. 22
  - Now Maestro Luca dal Borgo, a friar of S. Francis, who wrote about the regular geometrical bodies, was his pupil; and when Piero, after having written many books, grew old and finally died, the said Maestro Luca, claiming the authorship of these books, had them printed as his own, for they had fallen into his hands after the death of Piero.

The part of perspective in *De divina proportione* (1509) is based on Piero della Francesca’s *De prospectiva pingendi* (c. 1474)

- Status: claims of plagiarism are relative
  - Acknowledged by Pacioli in the *De divina proportione* and the *Summa*
Claims of plagiarism (3)

- The *Queste e el libro che tracta di mercatantie et usanze de paese* (Summa 1494) is based on Giorgio Chiarini 1481.
- Status: no plagiarism
  - Tariff tables now considered common property of 15th century merchant culture
  - Similar tables existed before Chiarini
The geometrical part of the *Summa* (1494) is based on an abaco manuscript.


- “all the ‘geometria’ of the Summa, from the beginning to page 59v. (118 pages numbered in folios), is the transcription of the first 241 folios of the Codex Palatino 577”.

Status: plagiarism, no translation, no acknowledgement, but the problems appear in many other 15th century manuscripts.
Many parts on arithmetic and algebra in the *Summa* (1494) are copied from *abbaco* manuscripts.


- “Unmerited fame”
- “Comparing the later [Summa] with large handwritten treatises [abbaco ms..] shall give many surprises”

Status: appropriation, no plagiarism
Phases in development of a mathematics textbook

1. The medieval tradition (800–1202)
2. Abacus manuscripts (1307–1494)
3. The beginning of algebraic theory (1494–1539)
   - Extracting general principles
4. Algebra as a model of demonstration (1545–1637)
5. From problems to propositions (1608–1643)
6. Axiomatic theory (1657–1830)

1. The medieval tradition

- Problems as vehicles for rote-learning
- Example: Alquin’s *Propositiones ad acuendos juvenes*
  - Two men were leading oxen along a road, and one said to the other: “Give me two oxen, and I’ll have as many as you have.” Then the other said: “Now you give me two oxen, and I’ll have double the number you have.” How many oxen were there, and how many did each have?
- Rhetoric of master and student
  - Declamation of a problem and asking for an answer
  - Rhyme and cadence essential in memorization
- Also in Hindu algebra (Brāhmagupta, Mahāvira,..)
2. The abbaco tradition

- Problems as algebraic practice
- Continuous development before Fibonacci (1202) till after Pacioli (1494)
- Learning by problem solving
- Knowledge disseminated between master and apprentice (often in family relations)
- Texts present the rhetorical reformulation of a problem using a *cosa*.
- Elegance of the solution depends on a clever choice of the unknown(s) (Antonio de’ Mazzighi, c. 1380):
Abbaco algebra follows a rigid rhetorical structure

1. Problem enunciation
2. Choice of the rhetorical unknown
3. Manipulation of polynomials
4. Construction of an ‘equation’ solvable by a standard rule
5. Root extraction
6. Numerical test
Example 1

Earliest known abacus manuscript on algebra

Jacopo da Firenze, ms. Vat. Lat. 4862, f. 39v (1307)
1. Enunciation of the problem

- Someone makes two business trips. On the first he makes a profit of 12. On the second he wins in the same proportion and when he ends his trip he found himself with 54. I want to know with how much he started with.

2. Choice of the unknown

Uses a (modified of combined) unknown quantity of the problem as the rhetorical unknown

- Pose that one begins with one cosa.
- Poni che se movesse con una cosa.
3. Manipulating polynomials

Nel primo viaggio guadangniò 12, dunque, compiuto il primo viaggio si truova 1 cosa e 12, adunque manifestamente apare che d’ongni una cosa faegli 1 cosa e 12 nel primo viaggio. Adunque, se ogni una cosa fae una cosa e 12, quanto far`a una cosa e 12. Convienti multiprichare una cosa e 12 via una cosa e 12 e partire in una cosa. [f. 30v]. Una cosa e 12 via una cosa e 12 fanno uno cienso e 24 cose e 144 numeri, il quale si vuole partire per una cosa e deve venire 54. E perci`o multipricha 54 via una cosa, fanno 54 cose, le quali s’aguagliano a uno cienso e 24 cose e 144 numeri. Ristora ciaschuna parte, cio[è] di chavare 24 cose di ciaschuna parte.
And on the first trip he wins 12. Then completing his first trip he finds 1 cosa and 12.

It is then also manifest that for each cosa one obtains 1 cosa and 12 on the first trip. How much does this become in the same proportion after the second trip?

It is appropriate to multiply one cosa and 12 with one cosa and 12 which makes one censo and 24 cosa and 144 numbers, which will become 54.

And therefore multiply 54 with one cosa. Makes 54 cose, which is equal with one censo and 24 cose and 144 numbers.

Restore each part, therefore subtract 24 cose from each part.

\[
\begin{align*}
x + 12 & \\
\frac{x + 12 \cdot 54}{x(x + 12)} & \\
(x + 12)(x + 12) & \\
x^2 + 24x + 144 & \\
54x & \\
x^2 + 24x + 144 = 54x \\
x^2 + 144 = 30x
\end{align*}
\]
4. Constructing an ‘equation’

- You will have that 30 cose are equal to one censo and 144 numbers.

\[30x = x^2 + 144\]

- Averai che 30 cose sono iguali a uno cienso e 144 numeri.
Applying a cannonical recipe:

- Divide in one censo, which becomes itself. Then take half of the cose, which is 15. Multiply by itself which makes 225, subtract the numbers which are 144, leaves 81. Find its [square] root which is 9. Subtract it from half of the cose, which is 15. Leaves 6, and so much is the value of the cosa.

And if you want to prove this, do as such. You say that on the first trip one wins 12 and with the 6 one started with, one has 18. So that on the first trip one finds 18. Therefore say as such, of every 6 I make 18; what makes 18 in the same proportion? Multiply 18 with 18, makes 324. Divide by 6, this becomes 54, and it is good.

Et se la voi provare, fa così. Tu di’ che nel primo viaggio guadagnio 12 et con 6 se mosse a 18. Siché nel primo viaggio se trovò 18. E però di’ così, se de 6 io fo 18, que farò de 18 a quella medesema ragione? Multipricha 18 via 18. Fa 324. Parti in 6, che ne vene 54, et sta bene.
Problems for generating algebraic theory
Extracting general principles from practice
Transformation of rhetoric of problem solving
  Solved problems become theorems
Case 1
  Pacioli *Summa* 1494 (numbers in continuous proportion)
  Taken from Antonio de’ Mazzinghi (c.1390)
Make three parts of 13 in continuous proportion so that the first multiplied with [the sum of] the other two, the second part multiplied with the [sum of the] other two, the third part multiplied with the [the sum of the] other two, and these sums added together makes 78.
Arrighi 1967, p. 15: “Fa’ di 19, 3 parti nella proporzione chontinua che, multiplichato la prima chontro all’altra 2 e lla sechonda parte multiplichato all’altra 2 e lla terza parte multiplichante all’altra 2, e quelle 3 somme agunite insieme faccino 228. Adimandasi qualj sono le dette parti”.

Make three parts of 19 in continuous proportion so that the first multiplied with [the sum of] the other two, the second part multiplied with the [sum of the] other two, the third part multiplied with the [the sum of the] other two, and these sums added together makes 228. Asked is what are the parts.
The problem in modern symbolism:
- \( \frac{x}{y} = \frac{y}{z}, x + y + z = a, x(y + z) + y(x + z) + z(x + y) = b \)

Expanding the product and summing the parts:
- \( 2xy + 2xz + 2yz = 228 \)

But as \( y^2 = xz \) this can be expressed as:
- \( 2xy + 2y^2 + 2yz = 228 \text{ or } 2y(x + y + z) = 228 \)

With the sum being 19, \( 2y \) thus equals 12 or \( y = 6 \)

The problem reduces to dividing 13 into two parts with 6 as the middle term, leading to:
- \( x^2 + 36 = 13x \)
This can be solved using the fourteenth key. Which says that you have to divide the sum of these multiplications, thus 78, by the double of 13. And this 13 is the sum of these quantities, which will give you the second part. Thus divide 78 by 26 gives 3 for the second part.
Problem

Formulation of general key 14

“On three quantities in continuous proportion, when multiplying each with the sum of the other two and adding these products together. Then divide this by double the sum of these three quantities and this always gives the second quantity”.

\[
y = \frac{x(y + z) + y(x + z) + z(x + y)}{2(x + y + z)} = \frac{b}{2a}
\]
Pacioli constructing algebraic theory

- Summa, dist. 6, treatise 6, art 10–12
  - Three numbers in GP: 15 keys
  - Four numbers in GP: 8 keys
- Summa, dist. 6, treatise 6, art 14
  - Three and four numbers in GP: 29 of 35 problems
- Most problems taken from *Trattato di Fioretti*
  - same problem, same values
  - same problem, different values
  - variations on problems
Magl. Cl. XI. 119, Problem RAA303 (c. 1417)

- **Enunciation**: Fammi di 10 tali 2 parte che multipricata l’una contro all’altra faccia 16.
- **Solution**: Noi sappiamo che è 2 e 8 ma facciamo questo leggieri per intendere le più forti. Farai cos: pogniamo che quello numero fosse una cosa...
- **Test**: Esse la vuoi provare dirai radicie di 9 sie 3 agiugni sopra 5 sono 8...

Algebraic theory after Pacioli

Extended by Cardano (1539) chap. 42 and 51

- Three numbers in continuous proportion
  \[
y = \frac{x(y+z) + y(x+z) + z(x+y)}{2(x+y+z)}
\]

- Four numbers in continuous proportion
  \[
  \left(\frac{x+y+z+u}{x+u}\right) = \frac{x+z}{x+z-y} + \frac{y+u}{y+u-z}
  \]
Conclusion about plagiarism

- Pacioli borrowed a lot from the abbaco tradition
  - Antonio de’ Mazzinghi
  - Piero della Francesca
- Pacioli contributed to the teaching of algebra
  - Extracting theory from algebraic practice
  - Generalizing problems on numbers in GP
- The *Summa* is an important bridge between the closed manuscript tradition and the mathematics books of the 16th century
Case 2: The second unknown

- Method explained in the *Summa*, dist. 8, treat. 6, f. 148v
- cosa and quantita
Case 2: The second unknown

- Antonio de’ Mazzighi (c. 1380) was the first (after Fibonacci) to use the second unknown for solving problems
  - cosa and quantità
- Used in *Trattato di Fioretti*
- Also used in Palatino 573
De’ Mazzinghi (Pal. 573 BNCF, f. 486r)
Solution with two unknowns

\[
\begin{align*}
\begin{cases}
a^2 + b^2 = 82 \\
\sqrt{a} + \sqrt{b} = 4 \\
a = x - \sqrt{y} \\
b = x + \sqrt{y}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
x^2 - 2x\sqrt{y} + y + x^2 + 2x\sqrt{y} + y = 82 \\
2x^2 + 2y = 82, \ y = 41 - x^2 \\
\begin{cases}
a = x - \sqrt{41 - x^2} \\
b = x + \sqrt{41 - x^2}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
a + b + 2\sqrt{ab} &= 16 \\
2x + 2\sqrt{x^2 - (41 - x^2)} &= 16 \\
2x + \sqrt{8x^2 - 164} &= 16 \\
16 - 2x &= \sqrt{8x^2 - 164} \\
256 + 4x^2 - 64x &= 8x^2 - 164 \\
4x^2 + 64x &= 420 \\
x^2 + 16x &= 105 \\
x &= 5 \\
y &= 41 - 25 = 16 \\
a &= 1, \ b = 9
\end{align*}
\]
Find two numbers with the sum of their squares equal to 20 and their product equal to 8

Pacioli uses $x - y$ and $x + y$ for the numbers

(Pacioli 1494, f. 148v): “Dove ponesti ponere l’uno essere 1.co p. 1q a e l’altro 1 co m. & q a”

$$(x^2 + 2xy + y^2) + (x^2 - 2xy + y^2) = 20,$$
$$(x - y)(x + y) = x^2 - 2xy - y^2 = 8$$
Pacioli 1494, f. 192r: “E per via de queste quantita sorda quali li antichi chiamavano cose seconde: si solvano moltissime forte rasoni chi ben le manegeia in li aguagliamenti ma te conven sempre fare che la quantita resti sola da un lato e da l’altro sia che vole meno o piu che non sa caso tutto sera valuta dela quantita e reca sempre tutto a uno quantita”.

And by way of this *quantita sorda*, which the ancients called the second unknown: so they can solve much harder problems, handling the equations well, which they always fit in such a way that the unknown appears only at one side [of the equation] and the other more or less, so that they not know in all cases the values of all the unknowns and therefore they always bring everything to one unknown.
Case 3: use in linear problems

Anonymous, BNCF, Fond. prin. V.152, c. 1390

Men buying a horse (ox) of unknown price

Tre àno danari e vogliono chonperare una ocha e niuno di loro non à tanti danari che per sé solo la possa chonperare; or dice il primo agli altri due: se ciaschuno di voi mi desse il 1/3 de’ suo danari i’ chonprerei l’ocha. Dice il sechondo agli altri due: se voi mi date il 1/3 più 4 de’ vostri danari i’ chonperò l’ocha. Dice il terzo agli altri due: se voi mi date il 1/4 meno 5 de’ vostri danari i’ chon però l’ocha. Poi agiunsono insieme i danari ch’eglino aveva no tra tutti e tre e posonglì sopra la valuta del’ocha ella somma farà 176, adimandasi quanti danari aveva chatuno per sé e che valeva l'ocha (f. 177r )

\[
\begin{align*}
    a + \frac{1}{3}(b + c) &= d \\
    b + \frac{1}{4}(a + c) + 4 &= d \\
    c + \frac{1}{4}(a + b) - 5 &= d \\
    a + b + c + d &= 176
\end{align*}
\]
First use in linear problems

- Uses *chosa* an *ocha* as unknowns
- Uses algebra to the point of two expressions in two unknowns
- Finds value using double false position
- Other example:

\[
\begin{align*}
1 \quad (b + c) &= y - x \\
\frac{1}{3} \quad b + c &= 3y - 3x \\
a + b + c &= 3y - 2x \\
a + b + c + d &= 4y - 2x = 176 \\
\begin{cases}
7y &= 13x + 4 \\
4y &= 2x + 176
\end{cases}
\end{align*}
\]

\[
\begin{align*}
73x - 7y &= 664 \\
3x + 7y &= 552
\end{align*}
\]
Maestro Benedetto

- Not in the *Trattato di pratica d’arismetrica* (Siena)
- *Trattato d’Abacho* (c. 1460, 18 copies)
  - Six problems solved with second unknown
    - “Men find a purse” and “men buy a horse”
  - Uses *quantità* and *chavallo* or *borsa*
  - Sum of the shares as the first unknown
  - Resolves indeterminacy by way of the two unknowns:

\[ 29y = 17x \]
Piero della Francesca

- Trattato d'abaco (c. 1480)

- Three problems with the second unknown
  - Linear problem: uses a triangle superscript for the second unknown (c. 39v)
  - Linear problem: cavallo and cosa (c. 40r)
  - Problem GP: cosa and quantitá (c. 125v)
Chuquet *Triparty*, 1484

Used in linear problems in the *Appendice*
- Problems 71, 75, 77, 78, 79 (Marre, 1881)
  \[
  \begin{align*}
  a + 7 &= 5(b + c - 7) + 1 \\
  b + 9 &= 6(a + c - 9) + 2 \\
  c + 11 &= 7(a + b - 11) + 3
  \end{align*}
  \]
- Same problem in Fibonacci and Barthelemy
  - Proto-algebraic rule
Chuquet uses $1^2$ for the second unknown (f. 196v)

De la Roche: “Ceste regle est appellee La Regle de la quantite”
Accused of plagiarism by Marre (1880, 1881)
  ① Does not reproduce the method for the same problem
  ② Reorganizes the text and improves notation

The first to give a name and description of the method
  ① “La regle de la quantite”
Removes ambiguity by using $\rho$ and $Qtite$ for $x$ and $y$

“*It is therefore necessary that the second, third or fourth position should be a number different from $\rho$. Because when the numbers for the second, third and fourth positions are the same and indistinguishable from the numbers for $\rho$, or the other positions, this would lead to confusion*”
Luca Pacioli, *Summa*, 1494

- Uses second unknown in three ways:
  1. Reproduces M° Antonio’s problems on number in continuous proportion (pt. 1, distinction 6, treatise 6)
  2. General explanation of “quantita sorda ne li libri pratichi antichi e stata chiamata cosa seconda” (pt. 1, distinction 9, treatise 6)
  3. Uses *cavallo* as second unknown (pt. 1, distinction 9, treatise 8) (without realizing so?)
Use of the second unknown < 1540

Antonio De’ Mazzinghi (1380)
Fond. prin. V.152
Benedetto, 1440
A
della Francesca, 1480
Chuquet, 1484
Pacioli, 1494
Cardano, 1539
de la Roche, 1520

Arab sources?
Luca Pacioli, *Summa*, 1494 (2)

- Probably based on ms A
- Same method as Chuquet
  - First unknown $a$
  - Second unknown $b$
  - Sum of three expressed in $x$

- Same problem appears in Catalan writings
  - Barcelona Ms. 71 c. 1500 and Ventallor (1521)
  - Do not adopt the method of two unknowns

\[
\begin{align*}
  a + \frac{1}{2}(b+c) &= 50 \\
  b + \frac{1}{3}(a+c) &= 50 \\
  c + \frac{1}{4}(a+b) &= 50
\end{align*}
\]
By the 1470’s there was a consistent system for algebraic notation in one unknown using an “equation sign”
  ◦ Regiomontanus (c 1463)
  ◦ Piero della Francesca
  ◦ Luca Pacioli (1478)

This notation system was not fully transferred to print
Pacioli in manuscript

- 1478, Vat. Lat. 3129, about 600 pages

- Consistent symbolism throughout the text
- Example: fol. 236\(^v\)
- Symbols for equations using +, −, =, \(x\), \(x^2\)
  
  \[
  20 - x^2 = -39 + 20x - x^2
  \]
Pacioli in print (1494)

- “equations” in full words (*Summa* f. 149r)

  \[ \begin{array}{c}
  \text{Landro de censo.} \\
  \text{equale.} \\
  \text{a cia.} \\
  \text{a censo.} \\
  \text{a numero.} \\
  \text{a numero e censo.} \\
  \end{array} \]

- no other mathematical ligatures or symbols
Regiomontanus (c 1460)

\[
\frac{x}{10-x}, \frac{10-x}{x}
\]

\[
\frac{100-10x}{x^2-10x}
\]

\[
\frac{2x^2+100-20x}{10x-x^2}
\]

Nürnberg Cent. V 56c, f. 23
Hoc problema geometrico more absolvere non licuit hactenus, sed per arte rei et census id efficere conabimur

“This problem cannot be proven by geometry at this point, but we will endeavor to accomplish it by the art of algebra”
Regiomontanus
De Triangulis omnimodis

DE TRIANGULIS
LIB. II.

XII.

Data perpendiculari atq; basi, & proportione laterum cognitis, titruncq latus cognoscere.

Hoc problema geometrico more absolvere nō licuit haecuis sed per artē rei & cenfus id efficerē conabīmur. Habeat itaque triangulus a b g perpendicularum a d, & basīm b g cognitorum, proportione laterum a b & a g datae, quaerimus titruncq lorum cognoscere. Sit uterī gratia, proportio a b ad b g taneg 3 ad 5, ita ut latus b a sit brevis latera b g, quod demum euenit ut casum b d breviser catur d g nemo incipītur possit, sit ergo d e aequalis ipsī b d, deturq perpendicularis a d 3, & basīs b g 20 pedes, ponō lineam e g 2 res, ita, unde linea e b erit 20 demptis duabus rebus, & eius medietas b d 10 minus 1 res, re liqua uero d g erit 10 & una res. ducō b d in fē, productur 1 cenfus & 100, demptis 20 rebus, quibus addo quadrātum perpendicularis scilicet 25, colliguntur 1 cenfus & 125, demptis 20 rebus, item b g in fē, sunt 1 cenfus, 20 res & 100, quibus addo quadraatum perpendicularis 25, colliguntur 1 cenfus 20 res & 125, itc habeō duo quadrata linearum a b & a g, quorum proportio est ut 9 ad 25, duplicata scilicet proportio 3 ad 5, quae erat, proportio laterum, cum itaque proportio quadratūm primi ad quadratum secundum sit taneg 9 ad 25, si duexero 25 in quadratum primum, itemō g 9 in quadratum secundum, quae productur erunt aequalis, resĭ tuos trans grind at absqid defecit, & ablatis aequalibus, utrobusque perducemur ad 16 cenfus & 2000 aequalis 680 rebus; quam omnequod resĭ tuos, pracepta artis docebūt. Lineā ergō g e quam posuì 2 res nota redundaŭt, hinc residua ex basi b g & eius medietas b d, quae cum perpendiculari a d, latus a b notum sufficiatūr, unde tamen d e & latus a g notum pronunciabitūr, quae libitin efficere.
Conclusion

- Question of plagiarism
  - not really meaningful in the context of late 15th cent
  - claims by Franci and Toti Rigatelli are tendentious

- Major contributions of Pacioli
  - restructuring knowledge from abbaco treatises
  - generalizing algebraic solutions in theory (keys)
  - didactical presentation of techniques (second unknown, double false position,..)
  - development of symbolism in the 15th century
  - gateway between manuscripts and print
  - very influential for 16th cent mathematics